VERY EASY PROOF OF THE BONDAREVA-SHAPLEY THEOREM

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Consider a TU game (N, v), where N is a finite set of players and $v : 2^N \to \mathbb{R}$ is a characteristic function. We call any collection $\mathcal{B}$ of coalitions *balanced* if

$$\exists(\lambda_S)_{S\in\mathcal{B}}\ \forall i \in N \sum_{\substack{S\in\mathcal{B}\\ S\ni i}} \lambda_S = 1$$

where $(\lambda_S)_{S\in\mathcal{B}}$ is a vector of positive weights. We call the game (N, v) *balanced* if for any balanced collection of coalitions $\mathcal{B}$ we have

$$\sum_{S\in\mathcal{B}} \lambda_S v(S) \leqslant v(N)$$

Theorem 1 (Bondareva-Shapley)**.** *(N, v) is balanced if and only if the core of (N, v) is non-empty.*

Proof. $(\Leftarrow)$: Suppose core is non-empty. Then

$$\exists(x_i)_{i\in N} \text{ such that } \sum_{i\in N} x_i = v(N) \text{ and } \sum_{i\in S} x_i \geqslant v(S)\ \forall S \subseteq N$$

Take any balanced collection of coalitions $\mathcal{B}$ with weights $(\lambda_S)_{S\in\mathcal{B}}$. Then

$$\begin{aligned}
\forall S \in \mathcal{B} \qquad \lambda_S \sum_{i\in S} x_i &\geqslant \lambda_S v(S) && \Rightarrow\\
\sum_{S\in\mathcal{B}} \sum_{i\in S} \lambda_S x_i &\geqslant \sum_{S\in\mathcal{B}} \lambda_S v(S) && \Rightarrow\\
\sum_{i\in N} \sum_{\substack{S\in\mathcal{B}\\ S\ni i}} \lambda_S x_i &\geqslant \sum_{S\in\mathcal{B}} \lambda_S v(S) && \Rightarrow\\
\sum_{i\in N} x_i = v(N) &\geqslant \sum_{S\in\mathcal{B}} \lambda_S v(S)
\end{aligned}$$

Thus, (N, v) is balanced.

$(\Rightarrow)$: By contraposition suppose that core is empty. Then two subsets of $\mathbb{R}^{|N|}$

$$\begin{aligned}
S_1 &= \{x \in \mathbb{R}^{|N|} \mid \sum_{i\in N} x_i = v(N)\}\\
S_2 &= \{x \in \mathbb{R}^{|N|} \mid \forall S \subsetneq N \ \sum_{i\in S} x_i \geqslant v(S)\}
\end{aligned}$$

are disjoint convex closed sets.

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Consider the linear programming problem

$$\min \sum_{i \in N} x_i$$
$$\text{subject to} \quad x \in S_2$$

It is equivalent to the following one

$$\min \sum_{i \in N} x_i$$
$$\text{subject to} \quad \sum_{i \in S} x_i \geqslant S \ \forall S \in 2^N \backslash \mathcal{C} = \mathcal{B}$$

where $\mathcal{C}$ is the collection of coalitions such that the corresponding inequalities are linearly dependent on the inequalities for $S \in \mathcal{B}$. Therefore, by construction of the problem, constraint qualification is satisfied. According to the Kuhn-Tucker theorem, there exists a vector of non-negative multipliers $(\lambda_S)_{S \in \mathcal{B}}$ such that FOC hold:

$$1 = \sum_{\substack{S \in \mathcal{B} \\ S \ni i}} \lambda_S \ \forall i \in N$$

and there exists a minimizer x^*, such that all inequalities are satisfied and

$$\lambda_S \sum_{i \in S} x_i^* = \lambda_S v(S) \ \forall S \in \mathcal{B}$$

Notice that coalitions from $\mathcal{B}$ cover N, since we cannot exclude all constraints with some player as linearly dependent. Then, from the equations above, it follows that

$$\sum_{S \in \mathcal{B}} \lambda_S v(S) = \sum_{S \in \mathcal{B}} \lambda_S \sum_{i \in S} x_i^* = \sum_{i \in N} \sum_{\substack{S \in \mathcal{B} \\ S \ni i}} \lambda_S x_i^* = \sum_{i \in N} x_i^* > v(N)$$

The last inequality follows from the fact that the hyperplane S_1 disconnects $\mathbb{R}^{|N|}$ and S_2 is unbounded (very big positive x are in it). Thus, for all $x \in S_2$ including x^* we should have $\sum_{i \in N} x_i > v(N)$. But this means that the game (N, v) is not balanced since $\mathcal{B}$ is the balanced collection of coalitions (by FOC), which violates the condition for game balancedness. ■

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