

**ECON 4113. HOMEWORK 2 (100 POINTS).**  
**DUE THURSDAY FEBRUARY 9 IN CLASS.**

In your answer you may skip long derivations. For example, when calculating the rank of a matrix you can just say what it is, without providing the calculations necessary to determine that. The same holds for the things you do with the systems of equations: just write the relationship between the variables that you think is important.

1. (30 points) Solve using the Lagrange Theorem

$$\begin{aligned} \max_{x,y,z \in \mathbb{R}} \quad & xyz + z \\ \text{s.t.} \quad & x^2 + y^2 + z \leq 6 \\ & x, y, z \geq 0 \end{aligned}$$

2. (30 points) Solve using the Lagrange Theorem

$$\begin{aligned} \max_{x,y \in \mathbb{R}} \quad & 3xy - x^3 \\ \text{s.t.} \quad & 2x - y = -5 \\ & 5x + 2y \geq 37 \\ & x, y \geq 0 \end{aligned}$$

3. (40 points) Facts: if  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is continuous, then the set  $\{x \in \mathbb{R}^n : f(x) \leq 0\}$  is closed and  $\{x \in \mathbb{R}^n : f(x) < 0\}$  is open. Also for any finite number of sets  $S_1, \dots, S_k$  it is true that: 1) if all  $S_i$  are closed then the intersection  $\cap_{i=1..k} S_i$  is closed; 2) if all  $S_i$  are open then the intersection  $\cap_{i=1..k} S_i$  is open.

Call  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  *strictly increasing* if

$$\forall x, y \in \mathbb{R}^n \quad \left[ x \geq y \Rightarrow f(x) > f(y) \right]$$

where  $x \geq y$  means that  $x_i \geq y_i$  for all  $i = 1..n$  and  $x_j > y_j$  for at least one  $j = 1..n$ .

Consider the problem

$$\begin{aligned} \max_{x \in \mathbb{R}^n} \quad & F(x) \\ \text{s.t.} \quad & G(x) \leq 0 \end{aligned}$$

where  $F : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $G : \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  $F$  is continuous and strictly increasing. Prove rigorously that if  $\Gamma = \{x \in \mathbb{R}^n : G(x) \leq 0\}$  is bounded then 1) the solution exists and 2) any solution  $\bar{x}$  is such that  $G^i(\bar{x}) = 0$  for at least one  $i \in \{1, \dots, m\}$  (the same as saying that  $\bar{x}$  is not *interior* solution).